Inflation, Exchange Rates and PPP in a Multivariate Panel Cointegration Model*

Tor Jacobson, Johan Lyhagen†, Rolf Larsson and Marianne Nessén

Abstract

New multivariate panel cointegration methods are used to analyze nominal exchange rates and prices in the four major economic powers in Europe, France, Germany, Italy and Great Britain for the post-Bretton Woods period. We test for PPP and find that the theoretical PPP relationship does not hold but there is a similar (1,-1.5,0.9 instead of 1,-1,1) relationship which is common for the investigated countries. Parametric bootstrap inference is used to deal with badly small sample sized tests.

Key words: Long-run purchasing power parity, multivariate cointegration analysis, bootstrap inference.

JEL Classification: F30, C15, C32.

*Comments by so and so are gratefully acknowledged.

†Corresponding author: Johan Lyhagen, Department of Economic Statistics, Stockholm School of Economics, 113 83 Stockholm, Sweden. *E-mail*: stjl@hhs.se.

1 Introduction

Does purchasing power parity hold in the long run? Are real exchange rates mean-reverting? A reading of the voluminous literature on this matter appears to give the following conclusions. If one applies unit root tests to real exchange rate data spanning long periods of time (say, close to a century or more) then evidence of long-run PPP is most often found (see e.g. Frankel 1986, Abuaf and Jorion 1990 and Lothian and Taylor 1996). However, when examining the recent post-Bretton Woods period of floating exchange rates the answer is less clear-cut. Conventional unit-root tests do not find evidence of PPP, while other approaches, e.g. using panel data, have provided evidence in favor of PPP[1].

In this paper we re-examine the case of PPP using long data sets, even though many would consider it a case closed. There are several reasons why we consider a re-examination warranted. First, earlier studies using long-run horizon data sets have typically analyzed the real exchange rate using various univariate techniques.[2] In contrast, we cast the analysis in terms of multivariate panel cointegration. The advantage of such a framework, as we see it, is described below. A second reason concerns size distortion, i.e. the erroneous rejection of a true null hypothesis due to an inappropriate asymptotic approximation. There are two sources for this. Firstly, Engel (1999) argues that the unit-root tests referred to above may have serious size biases due to the fact that any stationary process can be made arbitrary close to a nonstationary process. Secondly, as shown in Lyhagen (2000), using panel unit root test in the context of PPP gives invalid inference, i.e. the size of the test tends to one when the number of countries increases. This is due to that a common common trend is not considered when calculating the critical values. Both these effects leads to the false conclusion of a stationary real exchange rate. In this paper, asymptotic tests are augmented with parametric bootstrap analogues, whereby we reduce the effect, if not eliminate, the size distortion typically present in small-sample applications of asymptotic tests. As we bootstrap the multivariate model the problem of common common

[1]This interpretation of the post-Bretton Woods period is not self-evident. Cheung and Lai (1998), using more efficient unit-root tests, report evidence in favor of PPP. On the other hand, O'Connell (1998) provides a critical assessment of the evidence from panel studies.

[2]In addition to the references cited in the text, influential papers include Diebold, Husted and Rush (1991), Glen (1992) and Edison (1987).

trend is also solved.

We examine monthly data for the post-Bretton Woods years 1974-1999 for France, Germany, Italy and Great Britain, and the results of our analysis are the following. We do find evidence of cointegration between nominal exchange rates and prices; in fact the number of cointegrating vectors is exactly what PPP predicts. But the coefficients in the cointegrating vectors are not from what is compatible with PPP, although we find that all the cointegrating vectors are the same. Hence, we reject PPP.[3] We discuss this result in the concluding section, Section 5. Prior to that, Section 2 explains the implications of PPP in terms of cointegration while the asymptotics of the tests are in Section 3. Section 4 contains the cointegration analysis and in Section 5 there is a Monte Carlo simulation to investigate the small sample properties of the test statistics derived in Section 3. Description of the data used and proofs of the theorems are in the Appendix.

Multivariate framework — 'World-Wide PPP'

In contrast with most earlier studies of long-horizon data sets, we cast the analysis in terms of multivariate cointegration.[4] The multivariate nature of the framework offers two advantages. *First*, we are able to test for (bilateral) PPP between all countries in one system, meaning that the interdependent nature of the foreign exchange markets is taken into explicit account. Ideally, such an analysis should include prices and exchange rates of all large economies in order to fully account for the simultaneity. But doing so one would of course run into problems with degrees of freedom. Hence we have restricted the number of countries in the analysis to the four mentioned above, concentrating on what we believe to be major economies/currencies in Europe of the twentieth century. Furthermore, in this multivariate setup we will test not only individual bilateral PPP relations, but also whether all bilateral PPP relations hold simultaneously – i.e. 'world-wide' PPP. *Second*, nominal exchange rates and prices enter separately into the analysis. Hence

[3] Some would actually interpret our results as evidence of 'weak form' PPP; see e.g. MacDonald (1993). We prefer to associate PPP with the stricter requirement that the cointegrating relations satisfy certain linear restrictions. This is explained more in Section 2.

[4] Earlier cointegration studies using long-horizon data sets – Kim (1990) and Ardeni and Lubian (1991) – analyze nominal exchange rates and price ratios separately using the bivariate Engle-Granger two-step procedure.

no a priori restrictions are imposed on the joint behavior of prices and exchange rates (i.e. the so-called symmetry and proportionality conditions are not imposed, but instead subsequently tested for).[5]

Size and power issues in tests of long-run PPP

In the empirical PPP literature there has been much concern with issues of statistical power of the tests used when examining whether real exchange rates are mean-reverting (see e.g. Cheung and Lai 1998). On the other tack, Engel (1999) has shown that these tests may in fact have serious size biases when applied to random variables that contain a stationary but persistent component and a non-stationary component. On the panel unit root front Lyhagen (2000) have shown that the usually used critical values are wrong as they do not properly take care of the common common trend implied by PPP.

There is reason to believe that the usefulness of multivariate maximum likelihood cointegration analysis can be severely hampered by the curse of dimensionality, i.e. a large number of parameters in relation to a small number of observations. One undesirable effect is that the use of asymptotic critical values may jeopardize the validity of inference. This has been empirically verified in Jacobson, Vredin and Warne (1998). Gredenhoff and Jacobson (1998) have confirmed the presence and examined the nature of size distortion for likelihood ratio tests of linear restrictions on cointegrating vectors. However, they also found that parametric bootstrap testing is a robust alternative to asymptotic approximations, eliminating size distortions even for quite large systems and as few observations as 60. In this paper, all asymptotic tests (not only those of linear restrictions on cointegrating vectors) are augmented by parametric bootstrap analogues.[6]

[5] Earlier studies using the multivariate cointegration setup to analyze long-run PPP — Cheung and Lai (1993), Kugler and Lenz (1993), Johansen and Juselius (1992), MacDonald (1993) and Edison, Gagnon and Melick (1997) — have used data from the post-Bretton Woods period. Furthermore, these studies have examined PPP in series of trivariate systems (an exception is Nessén 1996). The typical result in these studies (and Nessén 1996) is that evidence of cointegration is found, but that the cointegrating relations fail comply with the restrictions implied by PPP.

[6] Edison et al. (1997) are also concerned about inappropriate use of asymptotic approximations in the context of multivariate maximum likelihood cointegration analysis of PPP. Analyzing post-Bretton Woods data they find only weak support for PPP, despite the use of small-sample critical values in the hypothesis testing.

2 PPP and linear restrictions on prices and exchange rates

We examine long-run PPP between four large European economies in a multivariate panel setting. The purpose of this section is to show how such a system is set up and to identify the restrictions implied by long-run PPP.

Denote the natural logarithm of the nominal dollar exchange rate of country i (that is, the number of currency i per unit dollar) by e_t^i . Further, let p_t^i be the natural logarithm of the price level in country i. Further, let p_t^* denote the price level in our numeraire country, the Great Britain. Define

$$X_{it} = \begin{bmatrix} e_t^i \\ p_t^i \end{bmatrix}$$

and then

$$X_t = \begin{bmatrix} e_t^1 \\ p_t^1 \\ \vdots \\ e_t^N \\ p_t^N \\ p_t^* \end{bmatrix}$$

where N is the number of countries except the base country, in our case four.

Now, if long-run bilateral PPP holds then the real exchange rates between all pairs of countries are stationary, or integrated of order 0, $I(0)$. This may be expressed as

$$q_t^i \equiv e_t^i - p_t^i + p_t^* \sim I(0) \qquad\qquad i = 1, ...N$$

where q_t^i is the real exchange rate between country i and the US. These N equations can be summarized as:

$$\begin{bmatrix} q_t^1 \\ q_t^2 \\ \\ \vdots \\ \\ q_t^N \end{bmatrix} \equiv \begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & \dots & 0 & 0 & 1 \\ \vdots & \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} e_t^1 \\ p_t^1 \\ \vdots \\ e_t^N \\ p_t^N \\ p_t^* \end{bmatrix} \sim I(0) \qquad (1)$$

It is easily recognized that the choice of base country is arbitrary. Premultiply the relationship with the matrix

$$\begin{bmatrix} 1 & 0 & \cdots & -1 & \cdots & 0 \\ 0 & 1 & & -1 & \cdots & 0 \\ \vdots & & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & -1 & \cdots & 0 \\ \vdots & & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & \cdots & 1 \end{bmatrix}$$

where the column of -1 is in the position of the new base country, gives the desired result. Note that the eigenvalues are $N-1$ ones and the last is minus one so the new relationships span the same space as the original one.

The equations in (1) can be evaluated in a vector error correction model on the form

$$\Delta X_t = \alpha\beta' X_{t-1} + \sum_{i=1}^{m-1} \Gamma_i \Delta X_{t-i} + \varepsilon_t. \tag{2}$$

Here, α and β are $N_{pq} \times Nr$, where $N_{pq} \equiv Np + q$ and β is given by

$$\beta = \begin{pmatrix} \beta_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ 0 & \dots & 0 & \beta_N \\ \beta_{N+1,1} & \cdots & & \beta_{N+1,N} \end{pmatrix}.$$

(This is a general formulation of the PPP example, where $p = 2$, $q = 1$ and $r = 0, 1, 2$ or $\alpha\beta'$ are of full rank.) No restrictions are imposed on the α, Γ_i $(N_{pq} \times N_{pq})$ and Ω $(N_{pq} \times N_{pq})$ matrices, the latter being the covariance matrix of ε_t $(N_{pq} \times 1)$. Assume that observations are taken at $t = 1, ..., T$.

The hypothesis of PPP implies that β is

$$\beta = \begin{bmatrix} \beta_{1x} & 0 & \cdots & 0 \\ \beta_{1p} & 0 & & 0 \\ 0 & \beta_{2x} & & 0 \\ 0 & \beta_{2p} & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & \beta_{Nx} \\ 0 & 0 & & \beta_{Np} \\ \beta_{1b} & \beta_{2b} & \cdots & \beta_{Nb} \end{bmatrix}$$

where rank of $\left[\beta_{ix}, \beta_{ip}, \beta_{ib}\right]$ is r_i. A simplification is to assume $r_i = r, i = 1, ..., N$. From this it is possible to test if the $[-1, 1, 1]$ restrictions is valid or the restrictions $\left[\beta_{ix}, \beta_{ip}, \beta_{ib}\right] = \left[\beta_x, \beta_p, \beta_b\right]$, i.e. the same parameters for all countries. Note that this model is similar to the one in Larsson and Lyhagen (1999) but with the addition of the last row which includes β_{ib} and the estimation procedures follows those outlined there.

The asymptotics of these test situations are considered in the next section.

3 Asymptotic results

Introduce the ECM

$$\Delta X_t = \alpha\beta' X_{t-1} + \sum_{i=1}^{m-1} \Gamma_i \Delta X_{t-i} + \mu + \varepsilon_t. \tag{3}$$

Here, $X_t \equiv \left(X'_{t,1}, ..., X'_{t,N}, X_{t,N+1}\right)'$ is $(Np+1) \times 1$ (the components $X_{t,i}$ are $p \times 1$ for $i = 1, ..., N$ and all $X_{t,N+1}$ are scalars) and α and β are $(Np+1) \times Nr$, where β is given by

$$\beta \equiv \begin{pmatrix} \beta_1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ 0 & \ldots & 0 & \beta_N \\ \beta_{N+1,1} & \cdots & & \beta_{N+1,N} \end{pmatrix}, \tag{4}$$

with blocks β_i that are $p \times r$ and $\beta_{N+1,i}$ that are $1 \times r$, $i = 1, ..., N$. The constant μ $((Np+1) \times 1)$ is not restricted to the cointegration space, i.e.

$\alpha'_\perp \mu \neq 0$. (This is a general formulation of the PPP example, where $p = 2$ and $r = 0, 1$ or 2.) No restrictions are imposed on the α, Γ_i $((Np+1) \times (Np+1))$ and Ω $((Np+1) \times (Np+1))$ matrices, the latter being the covariance matrix of ε_t $((Np+1) \times 1)$. Assume that observations are taken at $t = 1, ..., T$.

In the following, we will generalize the limit results for the tests worked out in Larsson and Lyhagen (1999) (in the sequel called LL) to the present situation. At first, we will try to see what Lemma 10.3 of Johansen (1995) looks like in this setting. For this, let

$$\begin{aligned} Z_{0t} &\equiv \Delta X_t, \\ Z_{1t} &\equiv X_{t-1}, \\ Z_{2t} &\equiv \left(\Delta X'_{t-1}, ..., \Delta X'_{t-m+1}, 1\right)', \\ \Psi &\equiv \left(\Gamma_1, ..., \Gamma_{m-1}, \mu\right), \end{aligned}$$

so that we may reformulate (3) in the more compact form

$$Z_{0t} = \alpha\beta' Z_{1t} + \Psi Z_{2t} + \varepsilon_t. \tag{5}$$

We need to look at the asymptotic behavior of Z_{1t} corrected for regression on Z_{2t}, for $t = [Tu]$ where $0 < u < 1$. To this end, we need a moving average representation of X_t, which is given in the following lemma. The lemma (and the subsequent theorems) relies on the assumption

Assumption A The roots to the characteristic equation corresponding to (3) have modulus > 1 or are equal to 1, and $\alpha'_\perp \Gamma \beta_\perp$ has full rank, where $\Gamma \equiv I_p - \sum_{i=1}^{m-1} \Gamma_i$.

This assumption guarantees that X_t is an $I(1)$ process (cf Johansen (1995), p. 49). The lemma is a reformulation of Grangers representation theorem, as given in theorem 4.2 of Johansen (1995).

Lemma 1 *If assumption A holds, we have the representation*

$$X_t = C\left(\mu t + \sum_{j=1}^{t} \varepsilon_j\right) + Y_t,$$

where Y_t *is* $I(0)$ *and* $C \equiv \beta_\perp \left(\alpha'_\perp \Gamma \beta_\perp\right)^{-1} \alpha'_\perp$ $((Np+1) \times (Np+1))$.

We now formulate the main limit result for $-2 \log Q_T$, where Q_T is the maximum likelihood ratio test of the hypothesis that $\operatorname{rank}(\alpha\beta') = Nr$, where β is as in (4), against $\operatorname{rank}(\alpha\beta') = Np + 1$.

The proof is a generalization of the corresponding theorem in LL. The main idea is to consider the three hypotheses H_3: rank $(\Pi) = Np+1$, H_2: $\Pi = \alpha\beta'$ where α and β are $(Np+1) \times Nr$ as above, but with no restrictions on β, and H_1: as H_2 but where β is as in (4). Then, $H_1 \subset H_2 \subset H_3$, and denoting the maximum likelihood ratio between H_i and H_j by Q_{ij} for $i < j$, we have $Q_T = Q_{13} = Q_{12}Q_{23}$, i.e.

$$-2\log Q_T = -2\log Q_{12} - 2\log Q_{23}.$$

The result will be that, as $T \to \infty$, $-2\log Q_{12}$ converges weakly to the χ^2 variate V, while $-2\log Q_{23}$ tends to U which has a Dickey-Fuller type distribution as given in the formulation of the theorem. Furthermore, $-2\log Q_{12}$ and $-2\log Q_{23}$ are asymptotically independent.

Theorem 2 *Under assumption A and if $\alpha'_\perp \mu \neq 0$ and $r > 0$, we have that as $T \to \infty$,*

$$-2\log Q_T \xrightarrow{w} U + V,$$

where, defining $\widetilde{W}(t)$ to be an $\{N(p-r)+1\}$-dimensional standard Wiener process (with mean zero and unity covariance matrix),

$$U = \operatorname{tr}\left\{\int d\widetilde{W}F'\left(\int FF'\right)^{-1}\int Fd\widetilde{W}'\right\},$$

and where V is χ^2 with $N(N-1)(p-r)r$ degrees of freedom, independent of U. The process F is $\{N(p-r)+1\}$-dimensional with components

$$F_i(u) \equiv \begin{cases} \widetilde{W}_i(u) - \int_0^1 \widetilde{W}_i(t)\,dt, & i = 1, \ldots, N(p-r), \\ u - \frac{1}{2}, & i = N(p-r)+1. \end{cases}$$

Our final object is to test if, given cointegrating rank $r = 1$ and $p = 2$, the cointegrating relation is $\left(\beta'_i, \beta_{N+1,i}\right) = c_i\left((-1,1),1\right)$ for all i and constants c_i. A more general formulation is that given r, $\left(\beta'_i, \beta'_{N+1,i}\right)' = H_i\psi_i$ for all i, where the H_i are known $(p+1)\times s$ matrices and the ψ_i are $s\times r$ and unknown. In the sequel, this hypothesis will be referred to as H_0. Our special case is given by $s = 1$, all $H_i = (-1,1,1)'$ and all $\psi_i = c_i$. The maximum likelihood ratio test of H_0 against H_1 (the hypothesis about restriction as in (4)) is denoted by Q_{01}.

Theorem 3 *Under assumption A and as $T \to \infty$, $-2\log Q_{01}$ is asymptotically χ^2 with $Nr\,(p-s+1)$ degrees of freedom.*

The proof is in the appendix.

The distribution of the test of common cointegrating space is treated in Larsson and Lyhagen (1999) where it is show that the test is χ^2 with $(N-1)\,r\,(p-r)+(N-1)\,r$ degrees of freedom. Note that the last $(N-1)\,r$ is due to the parameters of the base country which is not included in the analysis of Larsson and Lyhagen (1999).

4 The cointegration analysis

Our database contains monthly observations of wholesale prices and nominal exchange rates (vs the British pound) for Germany, France, Italy and Great Britain for the years 1974 - 2000, i.e. $N = 3$ and $T = 314$. The Appendix contains a fuller description of the data and sources, and also graphs, figures (1)-(3), of the exchange rate, real exchange rate and wholesale price series. A preliminary investigation concerning unit roots is carried out using the ADF test. The results are that wholesale prices and nominal exchange rates are non-stationary, further, the real exchange rates are also non-stationary if investigated on 5% level (the Italian is stationary on the 10% level).

The cointegration analysis performed in this paper employs methods developed by Johansen (1988, 1991). We begin by setting up the following vector error correction model (VECM):

$$\Delta X_t = \Gamma_1\,\Delta X_{t-1} + \ldots + \Gamma_{k-1}\,\Delta X_{t-k+1} + \Pi_k X_{t-k} + \mu + \delta D_t + \varepsilon_t \qquad (6)$$

where X_t (defined above) and μ are column vectors with seven elements, the Γ's and Π are matrices with coefficients, and ε_t is a Gaussian error term with zero mean and a covariance matrix Σ. The rank of Π is of central importance. If it has reduced rank less than $N*2+1$, then Π may be divided into two matrices α and β (i.e. $\Pi = \alpha\beta'$), where the matrix β contains the cointegrating vectors, i.e. $\beta'\,x_t$ is stationary.

In the subsequent sections we use this framework in the following way: First we estimate the number of cointegrating relations in a VECM of our seven-variable data set that satisfies standard specification tests. Second, we test hypotheses about the cointegration vectors. First we test if the cointegrating vectors span the same space and then if the theoretical relationship is within this space.

4.1 Specification and mis-specification analysis

The number of lags is specified using the information criterion proposed by Schwarz (1978) where a upper limit of five lags are pre-specified. The results suggest $k = 2$ would be appropriate. Given a lag of 2 the likelihood ration test of the three null $r = 0, 1, 2$ is calculated with the alternative of full rank. Instead of using the asymptotic distribution we use the method discussed above, i.e. a parametric bootstrap as it was used in Gredenhoff and Jacobson (1998). Note that data is generated under the null and with lags so the parameter uncertainty is dealt with to. A nominal size of 5% is used and the number of bootstrap replicates is 1000. The test statistics with the corresponding critical values are given in Table (2) . The null of $r = 0$ is rejected while the null of $r = 1$ is not, hence, we conclude that one cointegrating relationship per country is sufficient. The normalized cointegrating vectors are displayed in Table (3).

$$Table \ (2) \ in \ here$$

$$Table \ (3) \ in \ here$$

4.2 Testing linear restrictions

Having found support for the necessary condition for PPP, we now turn to the sufficient conditions. The multivariate setup used in this paper actually enables us to test for PPP in different ways. First, we test whether *all three* bilateral PPP relations span the same space, i.e. the four countries share the same economic laws but not necessarily the one outlined above. The test statistic is 17.4 with a bootstrapped critical value of 21.6 at a 5% nominal size, hence, we do not reject the null of a common cointegrating space. The normalized (with regards to β_{ix}) common cointegrating vector is $[1.00, -1.52, 0.885]'$ which have the corrects signs and does not seem to be far from the relationship implied of PPP. To test if PPP holds the likelihood ratio test with $[1, -1, 1]'$ as null is tested against common cointegrating space. The test statistic is 60.8 with a bootstrapped critical of 12.0, i.e. we reject the null.

In summary, we have found support for our hypothesis that the variables in x_t can be characterized by an error correction model like equation (6). This implies that they are driven by a limited number of common stochastic trends and therefore are tied together in the long run. There are three long-run, cointegrating, relations. However, none of these long-run relations can be interpreted in terms of PPP although the span the same space.

5 Small sample properties

Although we used Monte Carlo based inference in the empirical sections above it is of interest to show how well the asymptotic distributions works in small samples. To analyze this a Monte Carlo simulation is performed. The data generating process (DGP) is the empirical model estimated in the previous section. We are interested in five different null hypothesis. The first three considers the rank: $r = 0, r = 1, r = 2$, and the remaining two is tests on the cointegrating space: test of common space and test that the cointegrating vector is the theoretical PPP relationship, $1, -1, 1$. The alternative for the first three models are the usual full rank model and for the last two an unrestricted cointegrating model with rank one. For the very last model the alternative of a common cointegrating space is also considered. The largest eigenvalues of the DGP's are displayed in Table (4).

Table (4) *in here*

The Monte Carlo setup is as follows. First generate data according to the model under the null, then estimate the models under the null and the alternative and calculate the likelihood ratio statistic. Compare with the asymptotic critical value and note if the test reject or not reject the null. This is repeated 1000 times and the proportion of rejections are the size which should be compared to a nominal size of 5%. The size adjusted power, i.e. the simulated small sample critical values are used, of the tests are also of interest. For the null models $r = 0, r = 1, r = 2$ the DGP's are $r = 1, r = 2$ and full rank respectively. Regarding the cointegrating space tests the DGP is the $r = 1$ model. We also investigates the power when the null is the theoretical PPP but the data is generated from a model with common cointegrating space. The Monte Carlo simulation is done for sample sizes $T = 100, 200, 400, 800, 1600$ and 3200 and the number of replicates are 1000. The results are displayed in Table (5) and Table (6).

[Table (5) about here.]

[Table (6) about here.]

The results show that the well known problem in cointegration analysis that the for larger systems with many parameters the small sample critical values tends very slowly to the asymptotic (see e.g. Gredenhof and Jacobson (1998)). This result shows the very need for the use of the bootstrap or other size adjusting measures. The power properties are very satisfying for the larger sample sizes but not the smallest where we get the result that the power is less than the size. The most likely reason for this is that the critical values is quite dependent on where in the parameter space the DGP is. Note that as an iterative approach is used to estimate the cointegrating relations the cointegrating vectors for e.g. rank one do not have to be in the space spanned by the space of the cointegrating vectors for the model with two cointegrating vectors. This might also be an explanation. For the sample size closest ($T = 400$) to the one used in the empirical part the power properties is good.

6 Conclusions

Previous studies of long-run purchasing power parity have predominantly used univariate techniques (e.g. unit-root tests) and have often found support for long-run PPP. We, on the other hand, use a multivariate approach, and arrive at a different conclusion. We do find cointegrating vectors between nominal exchange rates and prices - and just the number that PPP would predict - but none of these can be interpreted in terms of PPP. An interesting result is that all the cointegrating vectors share the same space which indicates that the same economic law is valid for all four countries investigated, France, Germany, Italy and Great Britain.

It is difficult to reconcile the evidence given by traditional unit-root tests with the results provided in this study. What can explain this striking difference in conclusion? Possible explanations are offered by Engel (1999) (and Lyhagen (2000)), who argues that the traditional (panel) unit-root tests are greatly over-sized. The reliability of our results is enhanced by what we believe to be a well-specified statistical model and by the fact that all the asymptotic tests have been replaced by robust bootstrap inference.

Now, whereas the bootstrap test can be expected to be approximately correct in size, it should be noted that its power will not be higher, nor lower, than the power of a size-adjusted asymptotic test. This has been theoretically predicted for the general case by Davidson and McKinnon (1996) and verified for the likelihood ratio test of linear restrictions on cointegrating vectors by Gredenhoff and Jacobson (1998) using Monte Carlo simulation. Moreover, the results in Gredenhoff and Jacobson suggest that the power of the likelihood ratio test in a complex model based on relatively few observations, such as the one at hand, cannot be expected to be high. Despite this we do reject the null of PPP. Had we not, low test power could very well have driven that result. In other words, the bootstrap procedure ensures a proper size for the test and the insufficient power only strengthens the rejection result.

The conclusion arising from our analysis is that real exchange rates are non-stationary, even when examining data stretching over long periods of time. Hence shocks to real exchange rates do not subside with time, but instead have infinitely long-lived effects. This suggests that permanent real shocks are the predominant source of real exchange rate movements. A natural suggestion for future research is thus to develop models of real exchange rate behavior that focus mainly on real factors.

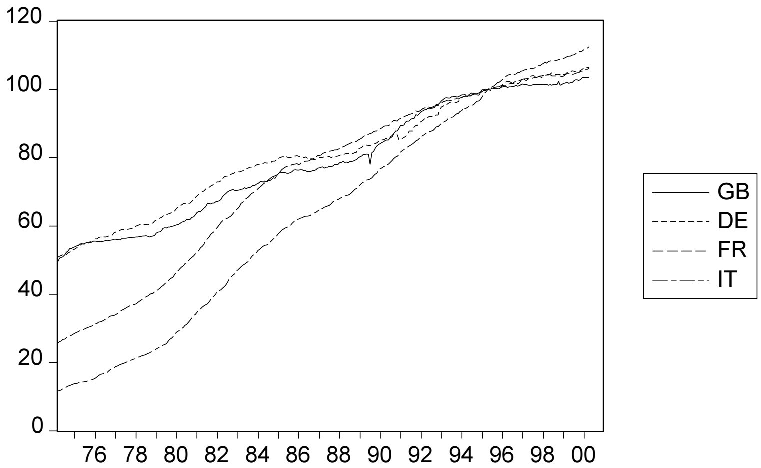

Figure 1: Monthly WPI for Great Britain, Germany, France and Italy

Appendix

6.1 Description of data

The database is comprised of three nominal exchange rates and four wholesale price indices. The frequency is monthly and the series run from 1974 to 1999. See Figures 1 -3. The exchange rates are the price of British pounds in German mark, French franc and Italian lire respectively. The WPI's are from row 63 in the IFS-tapes.

6.2 Proof of asymptotics

In the sequel, it will turn out to be convenient to use the reparametrisation $\beta' X_{t-1} = \varphi' \widetilde{X}_{t-1}$, where $\varphi \equiv \operatorname{diag}(\varphi_1, ..., \varphi_N)$ with $\varphi_i \equiv \left(\beta_i', \beta_{N+1,i}'\right)'$ for $i = 1, ..., N$, which are all $(p+1) \times r$, and

$$\widetilde{X}_{t-1} \equiv \left(X_{t-1,1}', X_{t-1,N}', X_{t-1,2}', X_{t-1,N}', ..., X_{t-1,N-1}', X_{t-1,N}'\right)',$$

which is $N(p+1) \times 1$. In other words, $\widetilde{X}_{t-1} = MX_{t-1}$, where the $N(p+1) \times$

Figure 2: Monthly exchange rates agains the british pound for Germany, France and Italy

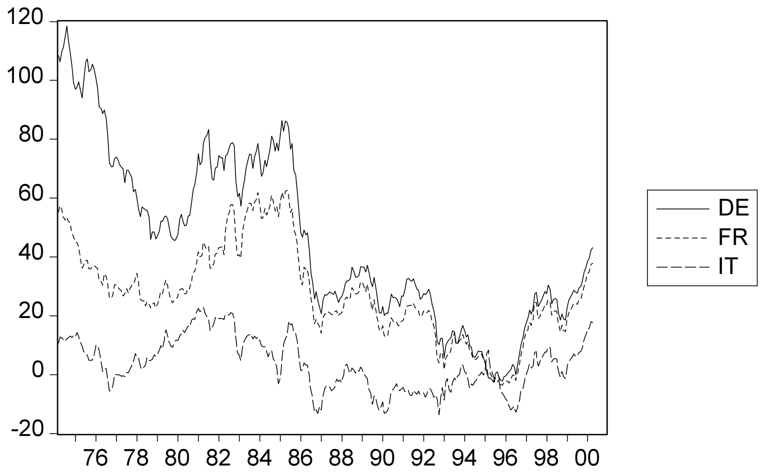

Figure 3: Monthly real exchange rate for Germany, France and Italy using Great Britain an base country.

$(Np+1)$ matrix M is given by

$$M \equiv \begin{pmatrix} \begin{pmatrix} I_p \\ 0 \end{pmatrix} & 0 & \cdots & 0 & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 & \begin{pmatrix} I_p \\ 0 \end{pmatrix} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \\ 0 & \cdots & 0 & \begin{pmatrix} I_p \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}.$$

If follows that $\beta' X_{t-1} = \varphi' M X_{t-1}$, i.e. $\beta = M'\varphi$, and we may re-write (5) as

$$Z_{0t} = \alpha\varphi' M Z_{1t} + \Psi Z_{2t} + \varepsilon_t. \tag{7}$$

Observe that the dominating deterministic term of the lemma has coefficient matrix

$$C\mu = \beta_\perp \left(\alpha'_\perp \Gamma \beta_\perp\right)^{-1} \alpha'_\perp \mu.$$

Since by assumption, $\alpha'_\perp \mu \neq 0$, we have $C\mu \neq 0$. Let $\tau \equiv (\tau'_1, ..., \tau'_N)' \equiv MC\mu \; (N(p+1) \times 1)$, where the τ_i are $(p+1) \times 1$ for $i = 1, ..., N$. Further, for each i, choose γ_i orthogonal to φ_i and to τ_i. Then, γ_i is $(p+1) \times (p-r)$ for $i = 1, ..., N$. Putting $\gamma \equiv \operatorname{diag}(\gamma_1, ..., \gamma_N)$, and $\overline{\gamma} \equiv \operatorname{diag}(\overline{\gamma}_1, ..., \overline{\gamma}_N)$ where $\overline{\gamma}_i \equiv \gamma_i \left(\gamma'_i \gamma_i\right)^{-1}$ for each i, and similarly for $\overline{\tau}$, it follows as in lemma 10.2 of Johansen (1995) that as a consequence of the lemma above, as $T \to \infty$ and for $0 < u < 1$,

$$\begin{aligned} T^{-1/2} \overline{\gamma}' M X_{[Tu]} &\xrightarrow{w} \overline{\gamma}' M C W(u), \\ T^{-1} \overline{\tau}' M X_{[Tu]} &\xrightarrow{w} u, \end{aligned}$$

because

$$\lim_{T \to \infty} T^{-1} [Tu] = u.$$

In other words, defining the norming matrix

$$\xi_T \equiv \begin{pmatrix} T^{-1/2} I_{N(p-r)} & 0 \\ 0 & T^{-1} \end{pmatrix},$$

which is $N_{pr} \times N_{pr}$ where $N_{pr} \equiv N(p-r) + 1$, we have the limit result

$$\xi_T \begin{pmatrix} \overline{\gamma}' \\ \overline{\tau}' \end{pmatrix} M X_{[Tu]} \xrightarrow{w} \begin{pmatrix} \overline{\gamma}' M C W(u) \\ u \end{pmatrix}, \tag{8}$$

where the latter matrix is $(Np+2)\times 1$. Furthermore, define

$$\overline{B} \equiv \begin{pmatrix} (\overline{\gamma}_1, \overline{\tau}_1) & 0 & \cdots & 0 \\ 0 & (\overline{\gamma}_2, \overline{\tau}_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & (\overline{\gamma}_N, \overline{\tau}_N) \end{pmatrix},$$

which is $N(p+1)\times N(p-r+1)$. Then,

$$\overline{B}' = \widetilde{M} \begin{pmatrix} \overline{\gamma}' \\ \overline{\tau}' \end{pmatrix},$$

where

$$\widetilde{M} \equiv \begin{pmatrix} \begin{pmatrix} I_{p-r} \\ 0 \end{pmatrix} & 0 & \cdots & 0 & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & & 0 & \\ 0 & \cdots & 0 & \begin{pmatrix} I_{p-r} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix},$$

which is $N(p-r+1)\times N_{pr}$. Furthermore, defining the $N(p-r+1)\times N(p-r+1)$ norming matrix

$$\widetilde{\xi}_T \equiv \begin{pmatrix} T^{-1/2}I_{p-r} & 0 & \cdots & 0 & 0 \\ 0 & T^{-1} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \\ & & & T^{-1/2}I_{p-r} & 0 \\ 0 & \cdots & & 0 & T^{-1} \end{pmatrix},$$

we have $\widetilde{M}\xi_T = \widetilde{\xi}_T\widetilde{M}$. Then, putting $\overline{B}_T \equiv \overline{B}\widetilde{\xi}_T$ and using

$$\overline{B}_T' = \widetilde{\xi}_T\overline{B}' = \widetilde{\xi}_T\widetilde{M}\begin{pmatrix} \overline{\gamma}' \\ \overline{\tau}' \end{pmatrix} = \widetilde{M}\xi_T\begin{pmatrix} \overline{\gamma}' \\ \overline{\tau}' \end{pmatrix},$$

left-hand multiplication of (8) by $\widetilde{M}$ yields

$$\overline{B}_T'MX_{[Tu]} = \widetilde{M}\xi_T\begin{pmatrix} \overline{\gamma}' \\ \overline{\tau}' \end{pmatrix}MX_{[Tu]} \xrightarrow{w} \widetilde{M}G_0(u), \tag{9}$$

where

$$G_0(u) \equiv \begin{pmatrix} \overline{\gamma}' MCW(u) \\ u \end{pmatrix},$$

which is $N_{pr} \times 1$. Further, define $G(u)$ to be $G_0(u)$ corrected for regression on 1, i.e.

$$G(u) \equiv \begin{pmatrix} \overline{\gamma}' MC \left\{ W(u) - \int_0^1 W(t)\, dt \right\} \\ u - \frac{1}{2} \end{pmatrix}. \tag{10}$$

In the following we will discuss the asymptotic distribution of the likelihood ratio test for cointegrating rank in the model (7).

To begin with, we will study the asymptotics of $\widehat{\varphi} - \varphi$. However, as in Johansen (1995), p. 179, we at first define $\widetilde{\varphi} \equiv \widehat{\varphi} \left(\overline{\varphi}' \widehat{\varphi} \right)^{-1}$ which, because of the decomposition $\widehat{\varphi} = \varphi \overline{\varphi}' \widehat{\varphi} + \overline{B} B' \widehat{\varphi}$, yields

$$\widetilde{\varphi} = \varphi + \overline{B} Y$$

where, because $B' \varphi = 0$,

$$Y \equiv B' \widehat{\varphi} \left(\overline{\varphi}' \widehat{\varphi} \right)^{-1} = B' \widetilde{\varphi} = B' \left(\widetilde{\varphi} - \varphi \right) \equiv \begin{pmatrix} Y_1 & 0 & \cdots & 0 \\ 0 & Y_2 & \ddots & \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & Y_N \end{pmatrix}, \tag{11}$$

where the diagonal blocks Y_i are $(p - r + 1) \times r$ for $i = 1, ..., N$. Moreover, we have $\widehat{\alpha} \widehat{\varphi}' = \widetilde{\alpha} \widetilde{\varphi}'$ where $\widetilde{\alpha} \equiv \widehat{\alpha} \widehat{\varphi}' \overline{\varphi}$. Finally, let $H_i^{(m)}$ be a $Nm \times m$ matrix of zeros except for the ith block where it is a unit matrix, i.e.

$$H_i^{(m)} \equiv \begin{pmatrix} 0 & \ldots & 0 & I_m & 0 & \ldots & 0 \end{pmatrix}'$$

and introduce the $N^2 r (p - r + 1) \times Nr (p - r + 1)$ matrix

$$K \equiv \begin{pmatrix} H_1^{(r)} \otimes H_1^{(p-r+1)} & \cdots & H_N^{(r)} \otimes H_N^{(p-r+1)} \end{pmatrix}.$$

Proof of Theorem 2: We will at first consider the asymptotics of $-2 \log Q_{12}$.

In the following, we will need some useful identities, to be found in e.g. Magnus and Neudecker (1988). For arbitrary matrices P, Q, R and S of dimensions such that the products below are defined, it holds that

$$
\begin{aligned}
\operatorname{tr}(PQ) &= \operatorname{tr}(QP), && (12) \\
(P \otimes Q)(R \otimes S) &= PR \otimes QS, && (13) \\
\operatorname{vec}(PQR) &= (R' \otimes P)\operatorname{vec} Q, && (14) \\
\operatorname{tr}(PQRS) &= (\operatorname{vec} S')'(R' \otimes P)\operatorname{vec} Q && (15) \\
(P \otimes Q)^{-1} &= \left(P^{-1}\right) \otimes \left(Q^{-1}\right) && (16)
\end{aligned}
$$

As in Johansen (1995), p. 91, concentrating out Z_{2t} leads us to the auxiliary regression

$$R_{0t} = \widetilde{\alpha}\widetilde{\varphi}' M R_{1t} + \widehat{\varepsilon}_t, \tag{17}$$

where R_{0t} and R_{1t} are Z_{0t} and Z_{1t} corrected for Z_{2t} and the $\widehat{\varepsilon}_t$ are independent normals, each with mean zero and covariance matrix Ω. For a moment, let us assume that Ω is fixed, the following arguments being applicable also when it is not, due to consistency. Then, apart from a constant, the log likelihood may be expressed as

$$\log L = -\frac{1}{2}\operatorname{tr}\left(\Omega^{-1}\sum_{t=1}^{T}\widehat{\varepsilon}_t\widehat{\varepsilon}_t'\right).$$

Furthermore, differentiating w.r.t. φ_i, $d\varphi' = \operatorname{diag}(0,...,0,d\varphi_i',0,...,0) = H_i^{(r)} d\varphi_i' H_i^{(p+1)\prime}$, we have $d\widehat{\varepsilon}_t = -\widetilde{\alpha}H_i^{(r)} d\varphi_i' H_i^{(p+1)\prime} M R_{1t}$, and it follows by (12) and (17) that

$$
\begin{aligned}
T^{-1} d\log L &= -T^{-1}\operatorname{tr}\left(\Omega^{-1}\sum_{t=1}^{T} d\widehat{\varepsilon}_t\widehat{\varepsilon}_t'\right) \\
&= T^{-1}\operatorname{tr}\left(H_i^{(p+1)\prime} M \sum_{t=1}^{T} R_{1t}\widehat{\varepsilon}_t'\Omega^{-1}\widetilde{\alpha}H_i^{(r)} d\varphi_i'\right) \\
&= \operatorname{tr}\left(H_i^{(p+1)\prime} M \widehat{S}_{1\varepsilon}\Omega^{-1}\widetilde{\alpha}H_i^{(r)} d\varphi_i'\right),
\end{aligned}
$$

where $\widehat{S}_{1\varepsilon} \equiv T^{-1}\sum_{t=1}^{T} R_{1t}\widehat{\varepsilon}_t'$. Moreover, since $\widehat{\varepsilon}_t' = R_{0t}' - R_{1t}' M' \widetilde{\varphi}\widetilde{\alpha}'$,

$$
\begin{aligned}
\widehat{S}_{1\varepsilon} &= S_{10} - S_{11} M' \widetilde{\varphi}\widetilde{\alpha}' \\
&= S_{1\varepsilon} - S_{11} M' (\widetilde{\varphi} - \varphi)\widetilde{\alpha}' - S_{11} M' \varphi\left(\widetilde{\alpha}' - \alpha'\right)
\end{aligned}
$$

with $S_{1\varepsilon} \equiv S_{10} - S_{11}M'\varphi\alpha'$ and $S_{ij} \equiv T^{-1}\sum_{t=1}^{T} R_i R_j'$ for $i, j = 0, 1$. Hence, using the consistency of $\widetilde{\alpha}$ (cf Johansen (1995), p. 181) and putting the derivative w.r.t. φ_i equal to zero, it follows that

$$H_i^{(p+1)\prime} M S_{1\varepsilon}\Omega^{-1}\alpha H_i^{(r)} = H_i^{(p+1)\prime} M S_{11} M' \left(\widetilde{\varphi} - \varphi\right)\alpha'\Omega^{-1}\alpha H_i^{(r)}, \tag{18}$$

for all i. Now, write $\overline{B}_T = \mathrm{diag}\left(b_1, ..., b_N\right)$, where $b_i \equiv \left(T^{-1/2}\overline{\gamma}_i, T^{-1}\overline{\tau}_i\right)$, which is $(p+1)\times(p-r+1)$. Then, for each i, it is easily seen that $b_i' H_i^{(p+1)\prime} = H_i^{(p-r+1)\prime}\overline{B}_T'$, and so, left-hand multiplication of (18) by b_i' yields

$$H_i^{(p+1)\prime}\overline{B}_T' M S_{1\varepsilon}\Omega^{-1}\alpha H_i^{(r)} = H_i^{(p+1)\prime}\overline{B}_T' M S_{11} M' \left(\widetilde{\varphi} - \varphi\right)\alpha'\Omega^{-1}\alpha H_i^{(r)}.$$

Moreover, inserting $\widetilde{\varphi} - \varphi = \overline{B}Y = \overline{B}_T Y_T$, where $Y_T \equiv \widetilde{\xi}_T^{-1} Y$, we get

$$H_i^{(p-r+1)\prime}\overline{B}_T' M S_{1\varepsilon}\Omega^{-1}\alpha H_i^{(r)} = H_i^{(p-r+1)\prime}\overline{B}_T' M S_{11} M' \overline{B}_T Y_T \alpha'\Omega^{-1}\alpha H_i^{(r)},$$

for $i = 1, ..., N$. To find $\mathrm{vec}\, Y_T$, we apply (14) to find

$$\begin{aligned} & \left(H_i^{(r)\prime} \otimes H_i^{(p-r+1)\prime}\right) \mathrm{vec}\left(\overline{B}_T' M S_{1\varepsilon}\Omega^{-1}\alpha\right) \\ = & \left(H_i^{(r)\prime} \otimes H_i^{(p-r+1)\prime}\right)\left(\alpha'\Omega^{-1}\alpha \otimes \overline{B}_T' M S_{11} M' \overline{B}_T\right) \mathrm{vec} Y_T, \end{aligned}$$

for $i = 1, ..., N$. This system may be rewritten as

$$K' v_T = K' C_T \mathrm{vec} Y_T, \tag{19}$$

with

$$\begin{aligned} v_T &\equiv \mathrm{vec}\left(\overline{B}_T' M S_{1\varepsilon}\Omega^{-1}\alpha\right), \\ C_T &\equiv \alpha'\Omega^{-1}\alpha \otimes \overline{B}_T' M S_{11} M' \overline{B}_T, \end{aligned}$$

with dimensions $N^2 r\left(p - r + 1\right) \times 1$ and $N^2 r\left(p - r + 1\right) \times N^2 r\left(p - r + 1\right)$, and where K is as defined previously.

In the case with no restrictions on φ, defining Z_T as the counterpart to Y_T in this case, we similarly get (see also LL)

$$\overline{B}_T' M S_{1\varepsilon}\Omega^{-1}\alpha = \overline{B}_T' M S_{11} M' \overline{B}_T Z_T \alpha'\Omega^{-1}\alpha,$$

i.e. using (14),

$$v_T = C_T\, \mathrm{vec}\, Z_T. \tag{20}$$

Now, because as $T \to \infty$, we have from (9) that

$$\overline{B}_T' M \left(X_{[Tu]} - T^{-1} \sum_{t=1}^{T} X_{t-t} \right) \xrightarrow{w} \widetilde{M} G(u), \tag{21}$$

so the limit of C_T, C say, which is also the asymptotic covariance matrix of v_T conditional on G, is given by, via (13),

$$C = \left(\alpha' \Omega^{-1} \alpha \right) \otimes \left(\widetilde{M} \int GG' \widetilde{M}' \right) = \left(I_{Nr} \otimes \widetilde{M} \right) J \left(I_{Nr} \otimes \widetilde{M}' \right), \tag{22}$$

where

$$J \equiv \alpha' \Omega^{-1} \alpha \otimes \int GG'.$$

(Note that C is singular.) Thus, by passing to the limit in (19) and (20), we find

$$\begin{aligned} K'v &= K'C\mathrm{vec}Y_T + o_P(1), & (23) \\ v &= C \,\mathrm{vec}\, Z_T + o_P(1), & (24) \end{aligned}$$

where v denotes the stochastic limit of v_T. Moreover, defining $a \equiv N^2 r (p - r + 1)$, $c \equiv Nr(p - r + 1)$ (note that $a \geqslant c$), the matrix dimensions are for K $a \times c$, for C $a \times a$, and for v, $\mathrm{vec}\, Y_T$ and $\mathrm{vec}\, Z_T$ $a \times 1$.

Now, as in LL,

$$\begin{aligned} K_\perp' \,\mathrm{vec}\, Y_T &= \begin{pmatrix} H_{1\perp}^{(r)\prime} \otimes H_1^{(p-r+1)\prime} \\ \vdots \\ H_{N\perp}^{(r)\prime} \otimes H_N^{(r)\prime} \end{pmatrix} \mathrm{vec}\, Y_T \\ &= \begin{pmatrix} \mathrm{vec}\left(H_1^{(p-r+1)\prime} Y_T H_{1\perp}^{(r)} \right) \\ \vdots \\ \mathrm{vec}\left(H_N^{(p-r+1)\prime} Y_T H_{N\perp}^{(r)} \right) \end{pmatrix} = 0, \end{aligned}$$

because $H_i^{(p-r+1)\prime}$ "picks out" the ith "block row" of Y_T and $H_{i\perp}^{(r)}$ picks out all "block columns" except for the ith one. Hence, the identity

$$I_a = K\overline{K}' + K_\perp \overline{K}_\perp'$$

yields $\operatorname{vec} Y_T = K\overline{K}' \operatorname{vec} Y_T$, so via (23),

$$K'v = K'CK\overline{K}' \mathrm{vec} Y_T + o_P(1). \tag{25}$$

Hence,

$$\mathrm{vec} Y_T = K\overline{K}' \operatorname{vec} Y_T = K(K'CK)^{-1} K'v + o_P(1). \tag{26}$$

Moreover, we will prove below that there is a full rank $a \times b$ matrix L such that

$$b \equiv Nr\{N(p-r)+1\} \leqslant a$$

and

$$\begin{aligned} L'_{\perp} \operatorname{vec} Z_T &= 0 + o_P(1), & (27) \\ L'\left(I_{Nr} \otimes \widetilde{M}\right) &= I_b. & (28) \end{aligned}$$

Hence, from (22), $L'CL = J$ and we can perform the same trick as above to obtain, via (24) left multiplied by L',

$$\mathrm{vec} Z_T = L\overline{L}' \operatorname{vec} Z_T = LJ^{-1}L'v + o_P(1).$$

Thus, in conjunction with (26),

$$\operatorname{vec} Z_T - \mathrm{vec} Y_T = Pv + o_P(1), \tag{29}$$

where we have the $a \times a$ matrix

$$P \equiv LJ^{-1}L' - K(K'CK)^{-1} K'. \tag{30}$$

Now, assume furthermore that, for some $b \times c$ matrix R, (observe that $c \leqslant b$), we have

$$K = LR, \tag{31}$$

i.e.

$$K'CK = R'L'CLR = R'JR.$$

Then, (30) becomes

$$\begin{aligned} P &= LJ^{-1}L' - LR(R'JR)^{-1} R'L' \\ &= L\left\{J^{-1} - R(R'JR)^{-1} R'\right\} L', & (32) \end{aligned}$$

Furthermore, we have the equality

$$J^{-1} - R\left(R'JR\right)^{-1}R' = J^{-1}R_{\perp}\left(R_{\perp}'J^{-1}R_{\perp}\right)^{-1}R_{\perp}'J^{-1}, \tag{33}$$

which is proved by noting that left-hand multiplication by $R'J$ or by $R_{\perp}'$ yields the same results on both sides, and so, (32) becomes

$$P = LJ^{-1}R_{\perp}\left(R_{\perp}'J^{-1}R_{\perp}\right)^{-1}R_{\perp}'J^{-1}L'. \tag{34}$$

Now, it follows as in LL that

$$-2\log Q_{12} = \operatorname{vec}\left(Z_T - Y_T\right)' C \operatorname{vec}\left(Z_T - Y_T\right) + o_P\left(1\right) = v'P'CPv + o_P\left(1\right), \tag{35}$$

where furthermore

$$\begin{aligned} P'CP &= LJ^{-1}R_{\perp}\left(R_{\perp}'J^{-1}R_{\perp}\right)^{-1}R_{\perp}'J^{-1}L'CLJ^{-1}R_{\perp}\left(R_{\perp}'J^{-1}R_{\perp}\right)^{-1}R_{\perp}'J^{-1}L' \\ &= LJ^{-1}R_{\perp}\left(R_{\perp}'J^{-1}R_{\perp}\right)^{-1}R_{\perp}'J^{-1}R_{\perp}\left(R_{\perp}'J^{-1}R_{\perp}\right)^{-1}R_{\perp}'J^{-1}L' \\ &= LJ^{-1}R_{\perp}\left(R_{\perp}'J^{-1}R_{\perp}\right)^{-1}R_{\perp}'J^{-1}L' = P. \end{aligned}$$

Hence, (35) yields

$$-2\log Q_{12} = v'Pv + o_P\left(1\right). \tag{36}$$

Moreover, conditional on G, $v'Pv$ is χ^2 distributed with the number of degrees of freedom equal to

$$\mathrm{E}\left(v'Pv\right) = \mathrm{E}\left\{\operatorname{tr}\left(Pvv'\right)\right\} = \operatorname{tr}\left\{P\,\mathrm{E}\left(vv'\right)\right\} = \operatorname{tr}\left(PC\right) = \operatorname{tr}\left(V^*\right),$$

and it is easily seen from (12) that

$$\begin{aligned} \operatorname{tr}\left(V^*\right) &= \operatorname{tr}\left(LJ^{-1}R_{\perp}\left(R_{\perp}'J^{-1}R_{\perp}\right)^{-1}R_{\perp}'J^{-1}L'C\right) \\ &= \operatorname{tr}\left(R_{\perp}'J^{-1}R_{\perp}\left(R_{\perp}'J^{-1}R_{\perp}\right)^{-1}\right) \\ &= \operatorname{tr}\left(I_{b-c}\right) = b - c. \end{aligned}$$

With $b = Nr\left\{N\left(p-r\right)+1\right\}$, this is

$$b - c = Nr\left\{N\left(p-r\right)+1-\left(p-r+1\right)\right\} = N\left(N-1\right)r\left(p-r\right).$$

Moreover, because the distribution of $v'Pv$ conditional on G does not depend on G, it also holds unconditionally that $v'Pv$ is χ^2 with $b-c$ degrees of freedom.

To conclude the discussion on the V component, it remains to prove (27), (28) and (31). To this end, note that in the representation (11) we may, for each i, introduce the partitions $Y_i = (Y_{i1}', Y_{i2}')'$, where Y_{i1} is $(p-r) \times r$ and Y_{i2} is $1 \times r$. Furthermore, note that under the restrictions on β hypothesis, the Z counterpart to Y may be partitioned accordingly. (The off-diagonal blocks of the Y_T counterpart, Z_T say, tend to 0 in probability as $T \to \infty$.) Moreover, introduce the $\{N(p-r)+1\} \times Nr$ matrix

$$\widetilde{Z} \equiv \begin{pmatrix} Z_{11} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ 0 & \cdots & 0 & Z_{N1} \\ Z_{12} & \cdots & & Z_{N2} \end{pmatrix}.$$

Now, (27) holds if the matrix L satisfies, denoting the limit of Z_T by Z,

$$\operatorname{vec} Z = L \operatorname{vec} \widetilde{Z}, \tag{37}$$

where L is $a \times b$, since $\operatorname{vec} Z$ is $a \times 1$ and $\operatorname{vec} \widetilde{Z}$ is $b \times 1$ with $b = Nr\{N(p-r)+1\}$. Moreover, it is easily seen that L may be taken as block diagonal with $Nr(p-r+1) \times r\{N(p-r)+1\}$ diagonal blocks L_i for $i = 1, ..., N$. For $i = 1$, we have

$$\operatorname{vec} \begin{pmatrix} Z_{11} \\ Z_{12} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = L_1 \operatorname{vec} \begin{pmatrix} Z_{11} \\ 0 \\ \vdots \\ 0 \\ Z_{12} \end{pmatrix}.$$

Here, in view of (14), we may write $L_1 = I_r \otimes \widetilde{L}_1$, where e.g.

$$\begin{pmatrix} Z_{11} \\ Z_{12} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \widetilde{L}_1 \begin{pmatrix} Z_{11} \\ 0 \\ \vdots \\ 0 \\ Z_{12} \end{pmatrix},$$

with

$$\widetilde{L}_1 \equiv \begin{pmatrix} I_{p-r} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & I_{p-r} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_{p-r} & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix},$$

which is $N(p-r+1) \times \{N(p-r)+1\}$ and of full rank. The corresponding equality for arbitrary i is similarly seen, and so, (37) is verified. Equation (28) is deduced by noting that

$$\begin{aligned} L'\left(I_{Nr} \otimes \widetilde{M}\right) &= \left(I_r \otimes \operatorname{diag}\left(\widetilde{L}'_1, ..., \widetilde{L}'_N\right)\right)\left(I_r \otimes \operatorname{diag}\left(\widetilde{M}, ..., \widetilde{M}\right)\right) \\ &= I_r \otimes \operatorname{diag}\left(\widetilde{L}'_1\widetilde{M}, ..., \widetilde{L}'_N\widetilde{M}\right) = I_b, \end{aligned}$$

with b as above, because as is easily seen, $\widetilde{L}'_i\widetilde{M} = I_{N(p-r)+1}$ for all i. To prove (31), at first observe that K may be seen as a block diagonal matrix with blocks K_i, $i = 1, ..., N$, where $K_i \equiv I_r \otimes \widetilde{K}_i$ where the $\widetilde{K}_i$ are $N(p-r+1) \times (p-r+1)$. For example, $\widetilde{K}_1 \equiv (I_{p-r+1}, 0, ..., 0)'$. Similarly, we may define R as block diagonal with blocks $R_i = I_r \otimes \widetilde{R}_i$ where the $\widetilde{R}_i$ are $\{N(p-r)+1\} \times (p-r+1)$. Hence, via (13), we need to choose the $\widetilde{R}_i$ so that $\widetilde{K}_i = \widetilde{L}_i\widetilde{R}_i$, which for $i = 1$ is seen to be true with

$$\widetilde{R}_1 \equiv \begin{pmatrix} I_{p-r} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The arguments for arbitrary i are similar.

As for $-2\log Q_{23}$, it follows as in Johansen (1995), p. 158-160 that it equals the sum of the roots ρ to the equation

$$\left|\rho\overline{B}'_T M S_{11} M'\overline{B}_T - \overline{B}'_T M S_{1\varepsilon}\alpha_\perp\left(\alpha'_\perp\Omega\alpha_\perp\right)^{-1}\alpha'_\perp S_{\varepsilon 1} M'\overline{B}_T\right| = 0.$$

In view of (9), we may rewrite this as

$$\left|\rho\widetilde{M}\int GG'\widetilde{M}' - \widetilde{M}\int GdW'\alpha_\perp\left(\alpha'_\perp\Omega\alpha_\perp\right)^{-1}\alpha'_\perp\int dWG'\widetilde{M}\right| = 0. \quad (38)$$

Now, because of the property $|AB| = |BA|$ for any matrices such that the products are well defined, we may factor out $\left|\widetilde{M}'\widetilde{M}\right|$ from the l.h.s. of (38), and since this factor is readily seen to be nonzero, (38) is equivalent to

$$\left|\rho \int GG' - \int GdW'\alpha_\perp \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1} \alpha_\perp' \int dWG'\right| = 0. \tag{39}$$

Further, following Johansen (1995), we define

$$W_1 \equiv \left(\overline{\gamma}' MC\Omega C'M'\overline{\gamma}\right)^{-1/2} \overline{\gamma}' MCW,$$

which is $N(p-r) \times 1$ and

$$W_2 \equiv \left\{\mu'\alpha_\perp \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1} \alpha_\perp' \mu\right\}^{-1/2} \mu'\alpha_\perp \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1} \alpha_\perp' W,$$

which is a scalar. Note that $N(p-r) < Np+1$, so the norming matrix $\overline{\gamma}' MC\Omega C'M'\overline{\gamma}$ is non-singular. Moreover, the vector $\widetilde{W} \equiv (W_1', W_2)'$ has dimension $N(p-r)+1$, which equals the dimension of $\alpha_\perp' W$ (α is $(Np+1) \times Nr$). Hence, the transformation from $\alpha_\perp' W$ to $\widetilde{W}$ is non-singular. Hence, we may insert $\widetilde{W}$ in place of $\left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2} \alpha_\perp' W$ in (39). Moreover, it is readily seen that $\widetilde{W}$ is a standard Wiener process, i.e. that its covariance matrix is the identity. For example, the covariance between W_1 and W_2 contains the factor $\mu'\alpha_\perp \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1} \alpha_\perp' \Omega C'M'\gamma = \mu' C'M'\gamma = \tau'\gamma$, which is zero by assumption. Similarly (cf (10)), we may insert $F \equiv (F_1', u - 1/2)'$ in place of G in (39), where $F_1 \equiv W_1(u) - \int_0^1 W_1(t)\, dt$. Hence, (39) becomes

$$\left|\rho \int FF' - \int Fd\widetilde{W}' \int d\widetilde{W}F'\right| = 0,$$

and the sum of of the $N(p-r)+1$ roots to this equation equals

$$\operatorname{tr}\left\{\int d\widetilde{W}F' \left(\int FF'\right)^{-1} \int Fd\widetilde{W}'\right\},$$

as asserted.

As for the independence between U and V, note that via (39),

$$\begin{aligned}
U &= \operatorname{tr}\left\{\left(\int GG'\right)^{-1} \int GdW'\alpha_\perp \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1} \alpha_\perp' \int dWG'\right\} \\
&= \operatorname{vec}\left(\int GdW'\alpha_\perp\right)' \left(\alpha_\perp' \Omega \alpha_\perp \otimes \int GG'\right)^{-1} \operatorname{vec}\left(\int GdW'\alpha_\perp\right),
\end{aligned}$$

see (15) and (16). Moreover, from (36), (21) and the definition of v, we have

$$\begin{aligned} V &= v'Pv \\ &= \operatorname{vec}\left(\widetilde{M}\int GdW'\Omega^{-1}\alpha\right)' P\operatorname{vec}\left(\widetilde{M}\int GdW'\Omega^{-1}\alpha\right) \\ &= \operatorname{vec}\left(\int GdW'\Omega^{-1}\alpha\right)'\left(I_{Nr}\otimes\widetilde{M}\right)' P\left(I_{Nr}\otimes\widetilde{M}\right)\operatorname{vec}\left(\int GdW'\Omega^{-1}\alpha\right) \end{aligned}$$

where the last equality follows from (14). Now, conditional on G, it is easily seen that $\int GdW'\Omega^{-1}\alpha$ and $\int GdW'\alpha_{\perp}$ are normally distributed and uncorrelated, hence independent. Thus, because P and C are constant conditional on G, U and V are conditionally independent given G. Furthermore, as was seen above, V is independent of G. Hence, denoting the simultaneous density of U and V by $f_{U,V}$, the density of G by f_G and the conditional densities by $f_{U|G}$, etcetera,

$$f_{U,V} = \int f_{U,V|G}f_G = \int f_{U|G}f_{V|G}f_G = f_V\int f_{U|G}f_G = f_Vf_U,$$

where the integrals are over the support of the G density. This shows the independence between U and V, and we are done. ■

Proof of Theorem 3: To find the asymptotic distribution of $-2\log Q_{01}$, we will utilize the decomposition

$$Q_{01}Q_{12} = Q'_{12}Q'_{22},$$

where Q'_{12} is the maximum likelihood ratio test of $H_0 \equiv H'_1$ against H'_2, where H'_2 is the restriction hypothesis $\varphi = H\psi$ for a not necessarily block diagonal φ. Moreover, Q'_{22} is the maximum likelihood ratio test of H'_2 against H_2, where H_2 is the general reduced rank hypothesis as above, i.e. the usual restriction test of Johansen (1995). Thus,

$$-2\log Q_{01} = -2\log Q'_{12} - 2\log Q'_{22} - (-2\log Q_{12}), \tag{40}$$

where we know $-2\log Q'_{22}$ from Johansen (1995) and $-2\log Q_{12}$ from the previous theorem and its proof. Hence, to find $-2\log Q_{01}$, we will need to establish $-2\log Q'_{12}$.

We will consider three different cases. Case 1 is when $s = r$, case 2 is $s = r + 1$ and case 3 is $s > r + 1$.

At first, let us consider cases 2 and 3. We will mimic the technique of the proof of the corresponding lemma in Johansen (1995), lemma 13.9. To this end, write $\varphi = H\psi$, where $H \equiv \text{diag}(H_1, ..., H_N)$ and $\psi \equiv \text{diag}(\psi_1, ..., \psi_N)$. Then, (5) becomes

$$Z_{0t} = \alpha\psi' H' Z_{1t} + \Psi Z_{2t} + \varepsilon_t,$$

and the counterpart to (17) is

$$R_{0t} = \widetilde{\alpha}\widetilde{\psi}' H' M R_{1t} + \widehat{\varepsilon}_t,$$

with tilde notation as above. Now, let $\tau_H \equiv (\tau'_{H1}, ..., \tau'_{HN})' = \overline{H}H'MC\mu$. Then, $\varphi'\tau_H = \psi' H' MC\mu = \varphi'\tau = 0$, i.e. φ is orthogonal to τ_H, which means that all φ_i are orthogonal to the corresponding τ_{Hi}. Then, choose $\gamma_H \equiv \text{diag}(\gamma_{H1}, ..., \gamma_{HN})$ such that for each i, γ_{Hi} $((p+1) \times (s-r-1))$ is orthogonal to φ_i and τ_{Hi} and such that $(\varphi_i, \tau_{Hi}, \gamma_{Hi})$ spans $\text{sp}(H_i)$. (In case 2, γ_{Hi} not defined, but the following arguments still follow with a slight modification.) Observe that γ_{Hi} is orthogonal to τ_i as well, since $H'_\perp \gamma_H = 0$ which implies $\gamma'_H \tau = \gamma'_H \overline{H}H'\tau = \gamma'_H \tau_H = 0$. Hence, for each i, $\gamma_{Hi} = \gamma_i \xi_{Hi}$ for some matrix ξ_{Hi}, where as before, γ_i is orthogonal to φ_i and τ_i. Consequently, we have, as $T \to \infty$ and for $0 < u < 1$,

$$\begin{aligned} T^{-1/2}\overline{\gamma}'_H M X_{[Tu]} &\xrightarrow{w} \overline{\gamma}'_H MCW(u), \\ T^{-1}\overline{\tau}'_H M X_{[Tu]} &\xrightarrow{w} u, \end{aligned}$$

or in other words, defining

$$\xi_{HT} \equiv \begin{pmatrix} T^{-1/2} I_{N(s-r-1)} & 0 \\ 0 & T^{-1} \end{pmatrix},$$

$$\xi_{HT} \begin{pmatrix} \overline{\gamma}'_H \\ \overline{\tau}'_H \end{pmatrix} M X_{[Tu]} \xrightarrow{w} \begin{pmatrix} \overline{\gamma}'_H MCW(u) \\ u \end{pmatrix}. \tag{41}$$

Now, replacing φ by ψ, M by $H'M$ and $H_i^{(p+1)}$ by $H_i^{(s)}$ throughout in the previous proof (observe that when differentiating w.r.t. ψ_i, $d\psi = H_i^{(r)} d\psi'_i H_i^{(s)\prime}$), we deduce the equality

$$H_i^{(s)\prime} H' M S_{1\varepsilon} \Omega^{-1} \alpha H_i^{(r)} = H_i^{(s)\prime} H' M S_{11} M' H \left(\widetilde{\psi} - \psi\right) \alpha' \Omega^{-1} \alpha H_i^{(r)}. \tag{42}$$

Next, let $\overline{B}_{HT} = \text{diag}(b_{H1}, ..., b_{HN})$, where $b_{Hi} \equiv \left(T^{-1/2}\overline{\gamma}_{Hi}, T^{-1}\overline{\tau}_{Hi}\right)$ $((p+1) \times (s-r))$. Then, because $(\varphi_i, \tau_{Hi}, \gamma_{Hi})$ spans $\text{sp}(H_i)$, we may for each i

write $b_{Hi} = H_i\eta_i$ for some $s \times (s-r)$ matrix η_i, so that $\overline{B}_{HT} = H\eta$ where $\eta \equiv \operatorname{diag}(\eta_1, ..., \eta_N)$. Further, left-hand multiplication of (42) by η_i' yields, because $\eta_i' H_i^{(s)\prime} H' = H_i^{(s-r)\prime}\eta' H' = H_i^{(s-r)\prime}\overline{B}_{HT}'$,

$$H_i^{(s-r)\prime}\overline{B}_{HT}' M S_{1\varepsilon}\Omega^{-1}\alpha H_i^{(r)} = H_i^{(s-r)\prime}\overline{B}_{HT}' M S_{11} M' H\left(\widetilde{\psi} - \psi\right)\alpha'\Omega^{-1}\alpha H_i^{(r)}.$$

Moreover, inserting $H\left(\widetilde{\psi} - \psi\right) = H\eta Y_{HT} = \overline{B}_{HT} Y_{HT}$, we get

$$H_i^{(s-r)\prime}\overline{B}_{HT}' M S_{1\varepsilon}\Omega^{-1}\alpha H_i^{(r)} = H_i^{(s-r)\prime}\overline{B}_{HT}' M S_{11} M' \overline{B}_{HT} Y_{HT}\alpha'\Omega^{-1}\alpha H_i^{(r)},$$

for $i = 1, ..., N$. To find $\operatorname{vec} Y_{HT}$, we apply (14) and (13) to find

$$\begin{aligned} & \left(H_i^{(r)\prime} \otimes H_i^{(s-r)\prime}\right) \operatorname{vec}\left(\overline{B}_{HT}' M S_{1\varepsilon}\Omega^{-1}\alpha\right) \\ = & \left(H_i^{(r)\prime} \otimes H_i^{(s-r)\prime}\right)\left(\alpha'\Omega^{-1}\alpha \otimes \overline{B}_{HT}' M S_{11} M' \overline{B}_{HT}\right) \operatorname{vec} Y_{HT}, \end{aligned}$$

for $i = 1, ..., N$. This system may be rewritten as

$$K_H' v_{HT} = K_H' C_{HT} \operatorname{vec} Y_{HT}, \tag{43}$$

with

$$\begin{aligned} v_{HT} & \equiv \operatorname{vec}\left(\overline{B}_{HT}' M S_{1\varepsilon}\Omega^{-1}\alpha\right), \\ C_{HT} & \equiv \alpha'\Omega^{-1}\alpha \otimes \overline{B}_{HT}' M S_{11} M' \overline{B}_{HT}, \end{aligned}$$

and

$$K_H \equiv \begin{pmatrix} H_1^{(r)} \otimes H_1^{(s-r)} & \cdots & H_N^{(r)} \otimes H_N^{(s-r)} \end{pmatrix},$$

which is $a_H \times c_H$, where $a_H \equiv N^2 r(s-r)$ and $c_H \equiv Nr(s-r)$. Then, it follows in the usual manner that $K_{H\perp}' \operatorname{vec} Y_{HT} = 0$, so that (43) implies

$$\operatorname{vec} Y_{HT} = K_H \overline{K}_H' \operatorname{vec} Y_{HT} = K_H \left(K_H' C_{HT} K_H\right)^{-1} K_H' v_{HT}. \tag{44}$$

Now, consider the hypothesis H_2': $\Pi = \alpha\beta'$ where β is *not* restricted as in (4), but where $\beta = M'\varphi = M'H\psi$, i.e. $\varphi = H\psi$. In this case, we have the equality

$$\overline{B}_{HT}' M S_{1\varepsilon}\Omega^{-1}\alpha = \overline{B}_{HT}' M S_{11} M' \overline{B}_{HT} Z_{HT}\alpha'\Omega^{-1}\alpha,$$

where Z_{HT} is the counterpart to Y_{HT}. Hence, we find as above that

$$v_{HT} = C_{HT} \operatorname{vec} Z_{HT}. \tag{45}$$

Furthermore, it is completely analogous to the previous proof (take s in place of $p+1$) to see that $K_H = L_H R_H$, $L'_{H\perp} \operatorname{vec} Z_{HT} = 0$, where L_H is $a_H \times b_H$ and R_H is $b_H \times c_H$ with $b_H \equiv Nr \{N(s-r-1)+1\}$. Moreover, as earlier, there is a $N(s-r) \times (N(s-r-1)+1)$ matrix

$$\widetilde{M}_H \equiv \begin{pmatrix} \begin{pmatrix} I_{s-r-1} \\ 0 \end{pmatrix} & 0 & \cdots & 0 & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & & 0 & \\ 0 & \cdots & 0 & \begin{pmatrix} I_{s-r-1} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix},$$

such that via (41) (cf (9)),

$$\overline{B}'_{HT} M X_{[Tu]} = \widetilde{M}_H \xi_{HT} \begin{pmatrix} \overline{\gamma}'_H \\ \overline{\tau}'_H \end{pmatrix} M X_{[Tu]} \xrightarrow{w} \widetilde{M}_H G_{H0}(u),$$

where, because $\overline{\gamma}_H = \gamma_H (\gamma'_H \gamma_H)^{-1} = \gamma \xi_H (\gamma'_H \gamma_H)^{-1} = \overline{\gamma} \gamma' \gamma \xi_H (\gamma'_H \gamma_H)^{-1}$ ($\xi_H \equiv \operatorname{diag}(\xi_{H1}, ..., \xi_{HN})$),

$$G_{H0}(u) \equiv \begin{pmatrix} \overline{\gamma}'_H MCW(u) \\ u \end{pmatrix} = \widetilde{\Phi}' G_0(u),$$

where

$$\widetilde{\Phi} \equiv \begin{pmatrix} \gamma' \gamma \xi_H (\gamma'_H \gamma_H)^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Similarly, letting $G_H(u) \equiv \widetilde{\Phi}' G(u)$, we find that the limit of C_{HT}, C_H say, satisfies, via (13),

$$\begin{aligned} C_H &= \left(I_{Nr} \otimes \widetilde{M}_H \widetilde{\Phi}'\right) \left(\alpha' \Omega^{-1} \alpha \otimes \int GG'\right) \left(I_{Nr} \otimes \widetilde{\Phi} \widetilde{M}'_H\right) \\ &= \left(I_{Nr} \otimes \widetilde{M}_H\right) \Phi' J \Phi \left(I_{Nr} \otimes \widetilde{M}'_H\right), \end{aligned}$$

where $\Phi \equiv I_{Nr} \otimes \widetilde{\Phi}$. Here, in analogy with the previous proof, $L'_H \left(I_{Nr} \otimes \widetilde{M}_H\right) = I_{b_H}$, and we have

$$L'_H C_H L_H = \Phi' J \Phi \equiv J_H, \tag{46}$$

and it also follows that (45) yields

$$\operatorname{vec} Z_{HT} = L_H J_H^{-1} L_H' v_H + o_P(1),$$

where v_H is the limit of v_{HT}. Together with (44), this implies

$$\operatorname{vec} Z_{HT} - \operatorname{vec} Y_{HT} = P_H v_H + o_P(1),$$

where

$$\begin{aligned} P_H &\equiv L_H J_H^{-1} L_H' - K_H \left(K_H' C_H K_H\right)^{-1} K_H' \\ &= L_H \left\{ J_H^{-1} - R_H \left(R_H' J_H R_H\right)^{-1} R_H' \right\} L_H', \end{aligned} \tag{47}$$

and it follows as in the previous proof that Q_{12}', the maximum likelihood ratio test of $H_0 \equiv H_1'$ against H_2', satisfies

$$-2 \log Q_{12}' = v_H' P_H v_H + o_P(1). \tag{48}$$

To show that $-2 \log Q_{01}$ is asymptotically χ^2, we will use (40) to express its limit as a positive definite quadratic form of normal variates. To this end, observe that defining $v_0 \equiv \operatorname{vec}\left(\int G dW' \Omega^{-1} \alpha\right)$, the limit result (21), its counterpart in the present setting, (14) and (13) imply

$$\begin{aligned} v_T &= \operatorname{vec}\left(\overline{B}_T' M S_{1\varepsilon} \Omega^{-1} \alpha\right) \xrightarrow{w} \operatorname{vec}\left(\widetilde{M} \int G dW' \Omega^{-1} \alpha\right) = \left(I_{Nr} \otimes \widetilde{M}\right) v_0, \\ v_{HT} &= \operatorname{vec}\left(\overline{B}_{HT}' M S_{1\varepsilon} \Omega^{-1} \alpha\right) \xrightarrow{w} \operatorname{vec}\left(\widetilde{M}_H \widetilde{\Phi}' \int G dW' \Omega^{-1} \alpha\right) \\ &= \left(I_{Nr} \otimes \widetilde{M}_H\right) \Phi' v_0, \end{aligned}$$

so that the corresponding limits v and v_H satisfy

$$L' v = v_0, \tag{49}$$

$$L_H' v_H = \Phi' v_0. \tag{50}$$

Hence, via (36) and (32),

$$\begin{aligned} -2 \log Q_{12} &= v' P v + o_P(1) \\ &= v' L \left\{ J^{-1} - R \left(R' J R\right)^{-1} R' \right\} L' v + o_P(1) \\ &= v_0' \left\{ J^{-1} - R \left(R' J R\right)^{-1} R' \right\} v_0 + o_P(1), \end{aligned} \tag{51}$$

and from the proof of theorem 13.9 of Johansen (1995), (15) and (13), we have the representation

$$
\begin{aligned}
& -2\log Q_{22}' \\
= \; & \operatorname{tr}\left\{ \left(\alpha'\Omega^{-1}\alpha\right)^{-1} \alpha'\Omega^{-1} \int dWG' \left(\int GG' \right)^{-1} \int GdW'\Omega^{-1}\alpha \right\} \\
& - \operatorname{tr}\left\{ \left(\alpha'\Omega^{-1}\alpha\right)^{-1} \alpha'\Omega^{-1} \int dWG'\widetilde{\Phi} \left(\widetilde{\Phi}' \int GG'\widetilde{\Phi} \right)^{-1} \widetilde{\Phi}' \int GdW'\Omega^{-1}\alpha \right\} \\
= \; & v_0' J^{-1} v_0 - v_0' \left\{ \left(\alpha'\Omega^{-1}\alpha\right)^{-1} \otimes \widetilde{\Phi} \left(\widetilde{\Phi}' \int GG'\widetilde{\Phi} \right)^{-1} \widetilde{\Phi}' \right\} v_0 \\
= \; & v_0' \left\{ J^{-1} - \Phi \left(\Phi' J \Phi\right)^{-1} \Phi' \right\} v_0.
\end{aligned}
$$

Moreover, from (47), (48) and (50),

$$
\begin{aligned}
-2\log Q_{12}' &= v_H' P_H v_H + o_P(1) \\
&= v_H' L_H \left\{ J_H^{-1} - R_H \left(R_H' J_H R_H\right)^{-1} R_H' \right\} L_H' v_H + o_P(1) \\
&= v_0' \Phi \left\{ J_H^{-1} - R_H \left(R_H' J_H R_H\right)^{-1} R_H' \right\} \Phi' v_0 + o_P(1).
\end{aligned}
$$

Hence, inserting into (40) and using (46),

$$
-2\log Q_{01} = v_0' S v_0 + o_P(1), \tag{52}
$$

where

$$
S \equiv R\left(R'JR\right)^{-1} R' - \Phi R_H \left(R_H' J_H R_H\right)^{-1} R_H' \Phi'.
$$

Now, observe that ΦR_H is $b \times c_H$. Assume that we may write

$$
\Phi R_H = R\Theta, \tag{53}
$$

where Θ is $c \times c_H$. (Recall that R is $b \times c$.) Then, we find as in the previous proof (cf (33)) that

$$
\begin{aligned}
S &= R\left\{ \left(R'JR\right)^{-1} - \Theta \left(\Theta' R' J R \Theta\right)^{-1} \Theta' \right\} R' \\
&= R\left(R'JR\right)^{-1} \Theta_\perp \left\{ \Theta_\perp' \left(R'JR\right)^{-1} \Theta_\perp \right\} \Theta_\perp' \left(R'JR\right)^{-1} R'.
\end{aligned}
$$

Hence, again as in the previous proof, (52) yields that $-2\log Q_{01}$ is asymptotically χ^2, where because of (49) and, taking the expectation conditional on G, $C = \mathrm{E}(vv')$, implying $\mathrm{E}(v_0 v_0') = L'CL = J$, the number of degrees of freedom equals, via (12),

$$\begin{aligned}
\mathrm{E}(v_0' S v_0) &= \mathrm{tr}\{S\,\mathrm{E}(v_0 v_0')\} = \mathrm{tr}(SJ) \\
&= \mathrm{tr}\left[R(R'JR)^{-1}\Theta_\perp\left\{\Theta_\perp'(R'JR)^{-1}\Theta_\perp\right\}\Theta_\perp'(R'JR)^{-1}R'J\right] \\
&= \mathrm{tr}\left[\Theta_\perp\left\{\Theta_\perp'(R'JR)^{-1}\Theta_\perp\right\}\Theta_\perp'(R'JR)^{-1}\right] \\
&= \mathrm{tr}(I_{c-c_H}) = Nr\{(p-r-1)-(s-r)\} \\
&= Nr(p-s-1),
\end{aligned}$$

which was to be shown.

To conclude the proof in cases 2 and 3, we must establish (53), which is equivalent to verifying that $R_\perp'\Phi R_H = 0$. Now, observe that ΦR_H is block diagonal with blocks $I_r \otimes \left(\widetilde{\Phi}\widetilde{R}_{Hi}\right)$, where we may write $\widetilde{\Phi} = \mathrm{diag}\left(\widetilde{\Phi}_1, ..., \widetilde{\Phi}_N, 1\right)$, each Φ_i being $(p-r)\times(s-r-1)$. Moreover, $R_\perp$ is block diagonal with blocks $I_r \otimes \widetilde{R}_{i\perp}$, so it is enough to show that $\widetilde{R}_{i\perp}'\widetilde{\Phi}\widetilde{R}_{Hi} = 0$ for all i. To see this for $i = 1$, note that

$$\begin{aligned}
&\widetilde{R}_{1\perp}'\widetilde{\Phi}\widetilde{R}_{H1} \\
= &\begin{pmatrix} 0 & I_{p-r} & 0 & \cdots & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ & \vdots & \ddots & & 0 & 0 \\ 0 & 0 & \cdots & 0 & I_{p-r} & 0 \end{pmatrix}
\begin{pmatrix} \widetilde{\Phi}_1 & 0 & \cdots & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & & 0 & \\ 0 & \cdots & 0 & \widetilde{\Phi}_N & 0 \\ 0 & \cdots & & 0 & 1 \end{pmatrix} \\
&\begin{pmatrix} I_{s-r-1} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \\
= &\ 0,
\end{aligned}$$

and the argument for a general i follows similarly.

Case 1 is equivalent to a test of a simple hypothesis on β, because if $s = r$, the ψ_i are all constants, so the space spanned by β is completely specified by

H. Now, from Johansen (1995), p.193, we have

$$\begin{aligned}
-2\log Q_{22}' &= \operatorname{tr}\left\{\left(\alpha'\Omega^{-1}\alpha\right)^{-1}\alpha'\Omega^{-1}\int dWG'\left(\int GG'\right)^{-1}\int GdW'\Omega^{-1}\alpha\right\} \\
&= v_0' J^{-1} v_0 + o_P\left(1\right),
\end{aligned}$$

as above. Moreover, in this case $-2\log Q_{12}' = 0$, because the block diagonal restriction does not involve any parameters. (With fixed $\beta = H$, ψ is a non-singular square matrix which may be absorbed into α.) Hence, if $s = r$, it follows via (51) and (40) that

$$-2\log Q_{01} = v_0' R\left(R' J R\right)^{-1} R' v_0 + o_P\left(1\right),$$

which by the usual arguments is asymptotically χ^2 with $c = Nr\left(p - r - 1\right) = Nr\left(p - s - 1\right)$ degrees of freedom, and we are done. ■

References

[1] Abuaf, N. and P. Jorion," Purchasing Power Parity in the Long Run" *Journal of Finance* 45, (1990).

[2] Ardeni, P. and D. Lubian, "Is there trend reversion in purchasing power parity?", *European Economic Review* 35, (1991).

[3] Cheung, Y. and K. Lai, "Long-Run Purchasing Power Parity During the Recent Float", *Journal of International Economics* 34, (1993).

[4] Cheung, Y. and K. Lai, "Mean reversion in real exchange rates", *Economics Letters* 46, (1994).

[5] Cheung, Y. and K. Lai, "Parity reversion in real exchange rates during the post-Bretton Woods period", *Journal of International Money and Finance* 17, (1998).

[6] Davidson, R. and J.G.McKinnon, "The Power of Bootstrap Tests", Discussion Paper no. 937, Queen's Institute for Economic Research, (1996).

[7] Diebold, F., S. Husted and M. Rush, "Real Exchange Rates under the Gold Standard", *Journal of Political Economy* 99, (1991).

[8] Doornik, J. A. and H. Hansen, "A Practical Test for Univariate and Multivariate Normality," manuscript, Nuffield College, (1994).

[9] Edison, H., "PPP in the Long Run: A Test of the Dollar/Pound Exchange Rate (1890 - 1978), *Journal of Money, Credit and Banking* 19, (1987).

[10] Edison, H., J. Gagnon and W. Melick, "Understanding the empirical literature on purchasing power parity: the post-Bretton Woods era", *Journal of International Money and Finance* 16, (1997).

[11] Engel, C., "Long-Run PPP May Not Hold After All", *Journal of International Economics,* 51, (2000).

[12] Frankel, J., "International Capital Mobility and Crowding-Out in the US Economy: Imperfect Integration of Financial Markets of or Goods Markets?" in R. Hafer, (ed.), *How Open is the US Economy?* (Lexington Books, 1986).

[13] Glen, J., "Real exchange rates in the short, medium, and long run", *Journal of International Economics* 33, (1992).

[14] Granger, C.W.J. and T. Teräsvirta, *Modelling Nonlinear Economic Relationships*, (Oxford University Press, 1993).

[15] Gredenhoff, M. and T. Jacobson, "Bootstrap Testing and Approximate Finite Sample Distributions for Tests of Linear Restrictions on Cointegrating Vectors", Sveriges Riksbank Working Paper Series no. 67, (April 1998).

[16] Jacobson, T., A. Warne and A. Vredin, "Are Real Wages and Unemployment Related?", *Economica* 65, (1998).

[17] Johansen, S., "Statistical Analysis of Cointegration Vectors." *Journal of Economic Dynamics and Control* 12, (1988).

[18] Johansen, S., "Estimation and Hypothesis Testing of Cointegration Vectors in Gaussian Vector Autoregressive Models." *Econometrica* 59, (1991).

[19] Johansen, S., *Likelihood-Based Inference in Cointegrated Vector Autoregressive Models*, (Oxford University Press, 1995).

[20] Johansen, S. and K. Juselius, "Maximum Likelihood Estimation and Inference on Cointegration — With Applications to the Demand For Money." *Oxford Bulletin of Economics and Statistics* 52, (1990).

[21] Johansen, S. and K. Juselius, "Testing Structural Hypotheses in a Multivariate Cointegration Analysis of the PPP and the UIP for UK." *Journal of Econometrics* 53, (1992).

[22] Kim, Y., "PPP in the Long Run: A Cointegration Approach." *Journal of Money, Credit and Banking* 22, (1990).

[23] Kugler, P. and C. Lenz, "Multivariate Cointegration Analysis and the Long-Run Validity of PPP", *Review of Economics and Statistics* 75, (1993).

[24] Larsson, R. and J. Lyhagen, "Likelihood-Based Inference in Multivariate Panel Cointegration Models", Working Paper Series in Economics and Finance, No: 331, Stockholm School of Economics, (1999).

[25] Lee, M.H., *Purchasing Power Parity*, (New York, Marcel Dekker Inc., 1976)

[26] Lothian, J. and M. Taylor, "Real Exchange Rate Behavior: The Recent Float from the Perspective of the Past Two Centuries" *Journal of Political Economy* 104, (June 1996)

[27] Lyhagen, J., "Why not use standard panel unit root test for testing PPP", Working Paper Series in Economics and Finance, No: 413, Stockholm School of Economics, (2000).

[28] MacDonald, R., "Long-Run Purchasing Power Parity: Is It For Real?", *Review of Economics and Statistics* 75, (1993).

[29] Mardia, K.V., "Measures of Multivariate Skewness and Kurtosis with Applications", *Biometrika* 57, (1970).

[30] McLeod, A.I. and W.K. Li,"Diagnostic Checking ARMA Time Series Models Using Squared Residual Autocorrelations", *Journal of Time Series Analysis* 4, (1983).

[31] Nessén, M., "Common Trends in Prices and Exchange Rates. Tests of Long-Run Purchasing Power Parity", *Empirical Economics* 21, (1996).

[32] O'Connell, P., "The Overvaluation of Purchasing Power Parity", *Journal of International Economics* 44, (1998).

	WPI			XRT				Real-XRT		
	DE	FR	IT	GB	DE	FR	IT	DE	FR	IT
ADF	-1.578	-2.134	-0.880	-1.08	-1.96	-1.69	-0.778	-2.07	-2.21	-2.61*
Lags	1	6	4	3	1	1	3	1	1	3

Table 1: ADF test where the lags been decided by testing down from 6 lags. A * denotes signoificant at the 10% level.

$r =$	0	1	2
Test stat.	217	69.5	0.467
Crit. value	135	70.8	7.31

Table 2: Test statistics and bootstrapped critical values (1000 replicates), a 5 % nominal size.

[33] Schwarz, G., "Estimating the Dimension of a Model", *Annals of Statistics* 6, (1978).

Authors affiliations

TOR JACOBSON, Research Department, Sveriges riksbank, 103 37 Stockholm, Sweden. *E-mail*: tor.jacobson@riksbank.se

ROLF LARSSON, Department of Information Science, Section for Statistics, Uppsala University, Box 513, 751 20 Uppsala, Sweden. *E-mail*: rolf.larsson@dis.uu.se.

JOHAN LYHAGEN, Department of Economic Statistics, Stockholm School of Economics, 113 83 Stockholm, Sweden. *E-mail*: stjl@hhs.se.

MARIANNE NESSÉN, Research Department, Sveriges riksbank, 103 37 Stockholm, Sweden. *E-mail*: marianne.nessen@riksbank.se.

	Germany	France	Italy
β_{ix}	1.00	1.00	1.00
β_{ip}	-5.12	9.87	-2.25
β_{ib}	0.671	-8.78	0.528

Table 3: Normalized unrestricted estimates of the cointegrating vectors

$r = 0$	1	2
1.00	1.00	1.00
1.00	1.00	0.991
1.00	1.00	0.974
1.00	1.00	0.974
1.00	0.991	0.896
1.00	0.966	0.890
1.00	0.966	0.890
0.827	0.403	0.455
0.398	0.403	0.407
0.301	0.242	0.302
0.288	0.242	0.302
0.288	0.281	0.262
0.203	0.281	0.218
0.156	0.141	0.218

Table 4: Absolute values of the eigenvalues of the compagnion matrix for $r = 0, 1, 2$

Null\T	100	200	400	800	1600	3200
$r = 0$	0.0037	0.0028	0.010	0.031	0.031	0.027
$r = 1$	0.000	0.001	0.001	0.002	0.006	0.013
$r = 2$	0.000	0.003	0.012	0.031	0.024	0.054
Common$\|r = 1$	0.000	0.000	0.001	0.008	0.025	0.034
PPP$\|r = 1$ vs unrestricted	0.650	0.363	0.182	0.101	0.072	0.056
PPP$\|r = 1$ vs common	0.926	0.705	0.298	0.127	0.072	0.057

Table 5: Size of PPP related panel tests

Null\T	100	200	400	800	1600	3200
$r = 0$	0.001	0.440	1.000	1.000	1.000	1.000
$r = 1$	0.054	0.389	0.997	1.000	1.000	1.000
$r = 2$	0.018	0.022	0.406	0.921	0.998	1.000
Common$\|r = 1$	0.050	0.251	0.936	1.000	1.000	1.000
PPP$\|r = 1$ vs unrestricted	0.101	0.485	0.994	1.000	1.000	1.000
PPP$\|r = 1$ vs common	0.083	0.662	1.000	1.000	1.000	1.000

Table 6: Size adjusted power of PPP related panel tests